

# Real Time Simulations in Lattice Gauge Theory

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I review the study of real (Minkowski) time correlators in hot, weakly coupled Yang-Mills theory via lattice methods. I concentrate on the Minkowski time topological susceptibility, which is related to the efficiency of baryon number violation at high temperature. It can be computed by approximating the IR fields as classical and solving their dynamics nonperturbatively on the lattice. However it is essential to include the UV degrees of freedom. Their influence can be computed perturbatively and included in the lattice model by the addition of auxiliary fields.

## 1. Introduction

Most of the work to date in lattice gauge theory has focused on computing Euclidean quantities. While this is a valid pursuit, there are very interesting questions to be asked about time-like separated correlators. For instance, almost all questions about dynamics involve such correlators. For QCD it appears at this time that we have no first principles nonperturbative tools to study these questions. However, the situation is somewhat different in the case of weakly coupled, hot Yang-Mills theory. This is a relevant question for the SU(2) sector of the standard model, for which the coupling is weak,  $\alpha_w \simeq 1/30$ . Naively we would think that perturbation theory should then work, and nonperturbative tools are not needed. However at very high temperatures this is not the case. High temperature questions are also interesting in cosmology; in particular we may need to understand the behavior of the SU(2) weak sector at high temperatures to understand the origin of the cosmological baryon number abundance.

## 2. Baryon number violation and topological susceptibility

First, a big question; how efficiently is baryon number violated in the standard model? To see that it is violated, recall what happens when there is an instanton in QCD: left handed quarks go in, right handed quarks go out (see Fig. 1).

Instantons are also permitted in SU(2) weak; the difference is that only left handed fermions

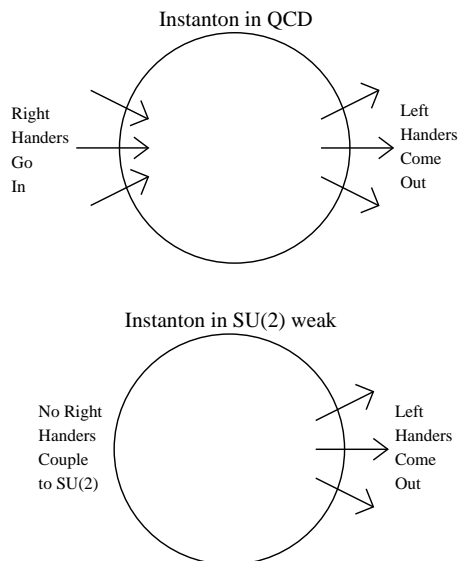


Figure 1. Top: cartoon of what an instanton in QCD does; it turns right handed quarks into left handed ones. But because SU(2) weak only couples to left handed fermions, a weak instanton (Bottom) creates a net particle number.

couple to SU(2) weak, and so, while left handers go out, no right handers go in. This is particle number violation; in particular, baryon number is violated, as first noted by t'Hooft [1].

So why don't we all die? Because the decay constant for baryons at normal temperatures is tiny. In vacuum we can relate the baryon number violation rate to the density of instantons in the Euclidean theory, giving a decay rate of order  $\exp(-16\pi^2/g_w^2) < 10^{-160}$ .

The situation may be different at high temperatures. Arnold and McLerran [2] have shown that the decay lifetime  $\tau$  for a baryon number excess is related to the Minkowski space topological susceptibility (often called the sphaleron rate),

$$\Gamma \equiv \int_{-\infty}^{\infty} dt \int d^3x \left( \frac{g_w^2}{32\pi^2} \right)^2 \times \langle F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a(x, t) F_{\alpha\beta}^b \tilde{F}_{\alpha\beta}^b(0, 0) \rangle, \quad (1)$$

through  $\tau^{-1} = (39/4T^3)\Gamma$ . Here  $\Gamma$  should be computed using the thermal density matrix but at zero chemical potential. This relation applies for temperatures  $T \gg m_{\text{proton}}$  and baryon number densities  $N_B/V \ll T^3$ .

Note that  $\Gamma$  is NOT the Euclidean topological susceptibility; Eq. (1) involves an integral over real (Minkowski) time. The two susceptibilities may be related in vacuum, but they are certainly not at high temperature; the Euclidean susceptibility is always exponentially small and falls with temperature, while  $\Gamma$  grows with increasing temperature, eventually becoming  $O(\alpha_w^5 T^4)$ .

To compute  $\Gamma$  we need a nonperturbative way to measure  $\int \langle F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a(x, t) F_{\alpha\beta}^b \tilde{F}_{\alpha\beta}^b(0, 0) \rangle$  at large  $T$ . Because the time integral involves all times, we cannot do this reliably by analytic continuation from Euclidean time. And as we shall now see, perturbation theory also proves unreliable.

### 3. Thermal Perturbation Theory and Classical Physics

My discussion here will follow [3]. One can arrange thermal perturbation theory in the real time formalism in terms of order symmetrized ( $F$ ) and antisymmetrized (retarded and advanced) propagators. Each diagram gives several contributions in which different propagators are  $F$  or advanced/retarded, and as we add loops to a diagram, the maximum possible number of  $F$  propagators increases by 1 per loop. In this formalism, temperature appears as the replacement on  $F$  propagators of the vacuum zero point fluctuation amplitude with the thermal fluctuation amplitude; for bosons,

$$\frac{1}{2} \rightarrow \frac{1}{2} + \frac{1}{\exp(E/T) - 1}. \quad (2)$$

Neglecting interactions  $E = \sqrt{k^2 + m^2}$ . At high temperatures the Higgs condensate, responsible for particle masses, dissolves and  $m \simeq 0$ . In the infrared, the Bose distribution function is large and we can make the expansion

$$\frac{1}{2} + \frac{1}{\exp(E/T) - 1} = \frac{T}{E} + \frac{1}{12} \frac{E}{T} - \frac{1}{720} \frac{E^3}{T^3} \dots, \quad (3)$$

which in the infrared is very well approximated by  $T/E$ . The same expansion for the fermions starts at  $O(E/T)$ , so their IR effects are negligible.

If we start out with the classical theory we get the same perturbation theory except that only diagrams with the maximum number of  $F$  propagators appear, with Eq. (3) simply replaced by  $T/E$ . Hence, the IR of a weakly coupled quantum theory at  $T \gg m$  behaves at leading order as a classical theory.

Moreover, this leads to a failure of perturbation theory. In vacuum, the loop counting expansion parameter is  $\alpha_w$ , which is small; but if one propagator per loop is an  $F$  propagator, the expansion parameter becomes  $(T/E)\alpha_w$ . If some bosonic mass scale is  $m \leq \alpha_w T$ , then for  $k \sim \alpha_w T$ , the expansion parameter is 1 and perturbation theory breaks down. However, perturbation theory only fails in the IR, and it fails only where the fields behave classically up to small corrections. Therefore we can treat the strongly coupled IR behavior by treating the IR degrees of freedom classically.

### 4. Classical Theory on the Lattice

With this in mind, Grigoriev and Rubakov conjectured that any unequal time IR dominated correlator in hot Yang-Mills theory would take the same value as its classical theory analogue, up to  $O(\alpha_w)$  corrections [4]. This turns out to be incorrect; it is essential to include the interactions between the IR modes and the (quantum but perturbative) UV modes [5]. But for starters we will “throw out” the UV and just look at  $\Gamma$  in the regulated classical theory.

We need a nonperturbative regulation which will preserve exact gauge invariance and remove the unwanted (nonclassical) UV degrees of freedom; naturally the lattice is the best candidate.

Real time classical Yang-Mills theory on the lattice looks almost like the molecular dynamics algorithm applied to the 3-D path integral without fermions. To each link matrix  $U_i \in \text{SU}(2)$  we associate a canonical momentum  $E_i$  in the Lie algebra, satisfying equations of motion [6]

$$\frac{dU_i}{dt} = i\tau^a E^a U_i, \quad (4)$$

$$\frac{dE_i}{dt} = -\sum_{\square} \frac{1}{2} \text{Tr} i\tau^a U_{\square}, \quad (5)$$

which arise from the Hamiltonian

$$H = H_{\text{Wilson, 3-D}} + \sum_{x,i} \frac{1}{2} E_i^a(x) E_i^a(x). \quad (6)$$

There are two differences from molecular dynamics. First, the  $E$  fields must satisfy Gauss' Law,

$$D_L \cdot E = 0, \quad (7)$$

which is conserved by the equations of motion but must be enforced on the initial conditions and considered in the thermalization. Second, we interpret  $t$  not as a “fake” time put in to perform a Monte-Carlo, but literally as Minkowski time.

To determine  $\Gamma$  we must measure the topology of the 4-D configuration gotten by taking 3-D configurations at successive times as neighboring 3-D slices of a 4-D lattice, see Fig. 2.

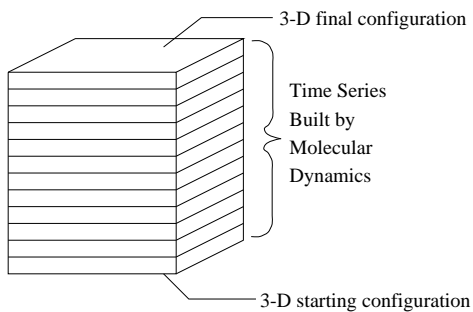


Figure 2. The 4-D space for which we must measure topology.

The space has open boundary conditions, so topology is technically undefined. Also we need a very efficient algorithm, since the algorithm which generates the configurations is very fast.

We fix the first problem by repeatedly appending a “new” initial and final 3-D slice, equal to the old one after some 3-D cooling. Sufficient cooling leads quickly to the vacuum (because we are in 3-D), and topology is defined with vacuum boundaries. The topological susceptibility is not affected in the limit of large molecular dynamics time. Finding an efficient algorithm is made easier because the gauge configurations are very smooth; first, the mean plaquette is much closer to  $\mathbf{1}$  than in the QCD setting because of the difference between 3-D and 4-D thermodynamics (not surprising as the underlying coupling is weak). Also, the temporal spacing is very small. Cooling within 3-D planes makes topological configurations spread in the 3 dimensions; they contract in the temporal direction, but it is highly resolved anyway and this is not a problem. It then remains to integrate  $F\tilde{F}$  with an  $O(a^2)$  improved operator. The details are in [7].

Is  $\Gamma$  in the classical theory lattice spacing independent? The data [8], presented in Fig. 3, says no.

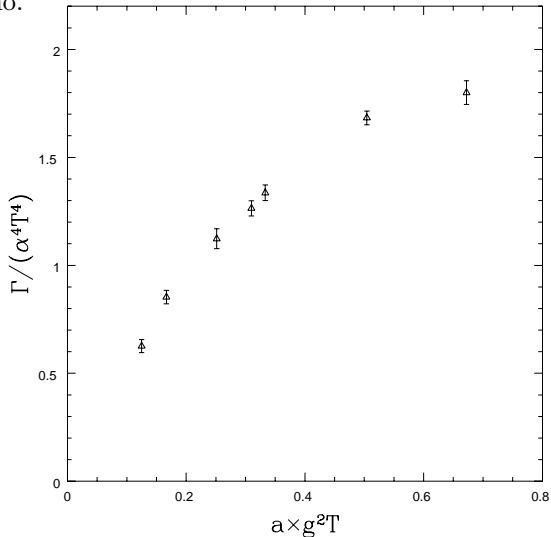


Figure 3. Topological susceptibility of hot lattice  $\text{SU}(2)$  Yang-Mills theory plotted against lattice spacing. Linear lattice spacing dependence is evident at the smallest lattice spacings.

This behavior is expected analytically [5]; it means that the UV must matter, and we should treat it properly.

## 5. Putting in the UV physics

We need to include properly the interactions between IR and UV. The UV does not behave classically; but it behaves perturbatively, and its leading order influence on the IR, the Hard Thermal Loops (HTL's), is known [9]. The HTL effective action is nonlocal [10], but it is equivalent to the effect of a large collection of classical, adjoint charged particles [11]. This means we may be able to incorporate the effects in a local way, and it provides the intuition of why we have to.

First, the intuition. The UV modes are short wavelength and so behave like classical particles. A large collection of charged particles is a plasma, and plasmas are conducting. By Lenz's Law, conducting media resist changes in magnetic fields. But evolution of IR magnetic fields is precisely what is required to get topology change. Hence, the more UV modes there are, the smaller  $\Gamma$  should be. This is basis of the arguments of Arnold, Son, and Yaffe [5,12].

How should we add the UV effects to the lattice IR theory? Two approaches seem reasonable; an N-body treatment and a Boltzmann-Vlasov treatment.

In the N-body approach, we add a large number of “particle” degrees of freedom to the lattice gauge theory [13]. Each particle has a coordinate, a momentum, and an adjoint representation charge. In Hu and Müller's formulation, the coordinates and momenta are each continuous degrees of freedom, so the particles “live” in the continuous space between the lattice sites, see Fig. 4.

The exact rules for the interactions between the particles and the lattice degrees of freedom were worked out in [14]. Each particle charge resides at the nearest lattice point, and the particle interacts with the lattice fields at the discrete instants when it passes from being nearest one site to being nearest another. The system has a conserved energy and phase space measure and exactly respects Gauss' Law; and in the small  $a$  and small particle charge limit it correctly reproduces the classical field + HTL effective theory.

The details of the Boltzmann-Vlasov approach appear in Kari Rummukainen's contribution to these proceedings [15], but I briefly review them

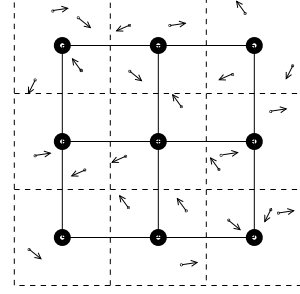


Figure 4. Cartoon of how “particle” degrees of freedom live in the continuous space between lattice sites. The charge resides at the vertex nearest the particle.

here. Rather than considering particles individually, one considers continuous particle population functions  $f(\vec{p}, \vec{x}, q)$ . Linearizing in the departure from equilibrium,  $f \rightarrow \delta f^a(\vec{p}, \vec{x})$ , which in the current setting can be further simplified by integrating over  $|p|$ , leaving only its direction  $\hat{p} \equiv \vec{v}$ . The resulting population function is called  $W^a(x, \vec{v})$ , and up to normalization it represents the net charge of particles moving in the  $\vec{v}$  direction at position  $x$ . The Yang-Mills-Boltzmann equations in continuum are [16]

$$(D_\nu F^{\nu\mu})^a = j_a^\mu, \quad (8)$$

$$j_a^\mu = m_D^2 \int \frac{d\Omega_v}{4\pi} v^\mu W^a(x, \vec{v}), \quad (9)$$

$$(v_\mu D^\mu)^{ab} W^b(x, \vec{v}) = v_\mu F_a^{0\mu}(x), \quad (10)$$

where  $v^\mu = (1, \vec{v})$  and  $d\Omega_v$  is the solid angle measure for integrating over  $\vec{v}$ .

The continuum equations not only need to be regulated on a lattice, discretizing space; they also need regulation on the sphere, which we do by expanding in spherical harmonics and truncating the series at a finite  $l_{\max}$ . For even  $l_{\max}$  there is surprisingly little truncation error [15].

In either approach, we expect the final answer to vary inversely with the conductivity, which is proportional to the Debye mass squared  $m_D^2$ . It makes sense, then, to fit  $\Gamma$  to the form

$$\Gamma = \kappa' \left( \frac{g^2 T^2}{m_D^2} \right) \alpha_w^5 T^4, \quad (11)$$

with  $\kappa'$  to be determined. The factor of  $(\alpha T)^4$  is

the four powers of the nonperturbative scale  $\alpha T$  required by dimensions and the last power of  $\alpha$  is the effect of the conductivity. The numerical data from the two approaches to the UV physics are in good agreement, as shown in Figure 5.

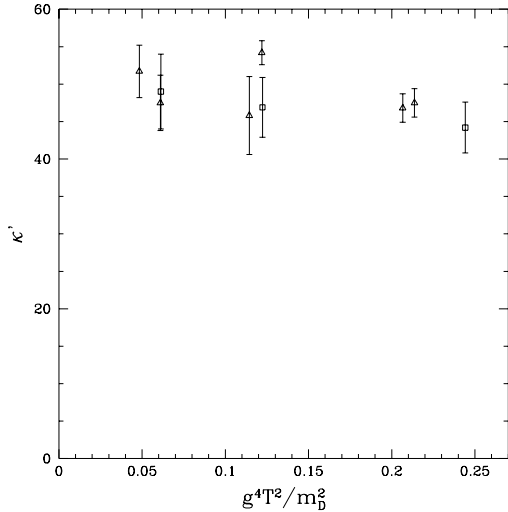


Figure 5.  $\Gamma$ , scaled by the expected parametric behavior, plotted against reciprocal conductivity. Theory predicts a flat line. Data with squares are using the N-body approach, triangles are with the Boltzmann-Vlasov approach.

The result is  $\kappa' \simeq 45$ , which is abundantly fast enough to erase any GUT generated initial baryon number (unless the GUT violates  $B-L$ ) and also fast enough to make electroweak baryogenesis viable if there is sufficient CP violation and loss of equilibrium at the electroweak phase transition.

## 6. conclusion

Real time techniques are available when the coupling is small. Although naively perturbation theory should be reliable in this case, it breaks down whenever some particle mass comes on order  $m \sim \alpha T$ ; but in this case the nonperturbative physics occurs only in the infrared, which is classical and can be treated nonperturbatively in real time on the lattice.

The influence of UV modes on the IR is very important in a gauge theory, and there are now

two different techniques, an N-body approach and a Boltzmann-Vlasov approach, to include it. The techniques are in good agreement.

The Minkowski topological susceptibility of pure Yang-Mills theory has been measured accurately, and this question is now closed. However there are still very interesting questions in nonperturbative IR physics at the electroweak phase transition which remain to be addressed, such as the dynamics of (Higgs) scalar condensates during the phase transition.

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